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BKP plane partitions

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Abstract: Using BKP neutral fermions, we derive a product expression for the generating function of volume-weighted plane partitions that satisfy two conditions. If we call a set of adjacent equal height-h columns, $h > 0$, an h-path, then *1.* Every h-path can assume one of two possible colours. *2.* There is a unique way to move along an h-path from any column to another.

Keywords: Integrable Hierarchies, Topological Strings.

Contents

1. Introduction

In [1], Okounkov and Reshetikhin observed that certain charged free fermion vertex operators can be used to generate plane partitions. In [2], with Vafa, they used this observation to compute the partition function of a topological string theory. As these vertex operators arise in KP theory [3], it is natural to look for analogous results in the context of other integrable hierarchies [4].

Definition 1. *An 'h-path' in a plane partition, or simply a 'path' when indicating the height h is not needed, is a set of adjacent equal height-h columns, where $h > 0$.*

In this note, we use BKP neutral free fermion vertex operators to obtain the generating function of volume-weighted plane partitions [5], that satisfy two conditions. *1.* Every h-path, $h > 0$, can assume one of two possible colours, so it contributes a factor of 2 to the multiplicity of the plane partition, irrespective of h. *2.* There is a unique way to move along an h-path, from one column to another, or equivalently *'every h-path is 1-column wide'*.

6	6	3	2
5	4	3	1
3	3	3	

Figure 1: A tableau-like representation of a plane partition. There is a 3-path of length 5, a 6-path of length 2, and 4 different height paths of length 1 each. There is a unique way to move from any column on a path to another column. Counting a no-move on a length-1 h-path as the one (and only) possible move, every h-path is 1-column wide, so it qualifies as a BKP partition.

Definition 2. *A 'BKP plane partition' is a plane partition that satisfies the above two conditions.*

Example 1. *In figure 1, we use tableau-like notation to represent a plane partition of the type counted in this note. The integers are the column heights. The volume of a plane partition is the sum of all column heights. The volume in this case is 39. There are 6 h-paths. Each path can assume one of two possible colours, so the multiplicity of this plane partition is* $2^6 = 64$.

2. BKP fermions

In this section, we review basic facts related to BKP neutral fermions [4].

2.1 Neutral fermions

Following [4], we consider the neutral fermion field $\Phi(k) = \sum_{m\in\mathbb{Z}} \phi_m k^m$, where the mode operators, ϕ_m, satisfy the anti-commutation relation

$$[\phi_m, \phi_n]_+ = (-)^m \delta_{m+n,0}, \quad m, n \in \mathbb{Z} \tag{2.1}$$

2.2 Fock states

We indicate an initial Fock state by $\langle \ldots, i_2, i_1|$, with $\ldots < i_2 < i_1 \le 0$, and a final state by $|j_1, j_2, \ldots\rangle$, with $0 \le j_1 < j_2 < \ldots$ where, as usual, the integers $\{i_m, j_n\}$ indicate filled neutral fermion energy states.

- The action of ϕ_m, $m > 0$, is

$$\langle \ldots, i_2, i_1|\phi_{(m>0)} = \begin{cases} (-)^{m+k}\langle \ldots, i_{k+1}, -m, i_k, \ldots, i_1|, & i_{k+1} < -m < i_k \\ 0, & \textit{otherwise} \end{cases}$$

$$\phi_{(m>0)}|j_1, j_2, \ldots\rangle = \begin{cases} (-)^{m+k-1}|j_1, \ldots, j_{k-1}, j_{k+1}, \ldots\rangle, & m = j_k \\ 0, & \textit{otherwise} \end{cases}$$

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- The action of ϕ_m, $m < 0$, is

$$\langle \ldots, i_2, i_1|\phi_{(m<0)} = \begin{cases} (-)^{k-1}\langle \ldots, i_{k+1}, i_{k-1}, \ldots, i_1|, & m = i_k \\ 0, & \textit{otherwise} \end{cases}$$

$$\phi_{(m<0)}|j_1, j_2, \ldots\rangle = \begin{cases} (-)^{k}|j_1, \ldots, j_k, -m, j_{k+1}, \ldots\rangle, & j_k < -m < j_{k+1} \\ 0, & \textit{otherwise} \end{cases}$$

- The action of ϕ_0 is

$$\langle \ldots, i_2, i_1|\phi_0 = \begin{cases} \frac{1}{\sqrt{2}}\langle \ldots, i_2, i_1, 0|, & i_1 \neq 0 \\ \frac{1}{\sqrt{2}}\langle \ldots, i_2|, & i_1 = 0 \end{cases}$$

$$\phi_0|j_1, j_2, \ldots\rangle = \begin{cases} \frac{1}{\sqrt{2}}|0, j_1, j_2, \ldots\rangle, & j_1 \neq 0 \\ \frac{1}{\sqrt{2}}|j_2, \ldots\rangle, & j_1 = 0 \end{cases}$$

2.2.1 Remark

Notice that 0 is an allowed filling number, which can be added or removed by the action of ϕ_0. In the following, this action is used to represent any Fock state in terms of an *even* number of mode operators acting on the vacuum.

2.3 Strict partitions

Definition 3. *A strict partition, $\widehat{\mu}$, is a partition that has only distinct parts. In this note, we take the number of parts to be always even by allowing for at most one part of length 0, which agrees with Remark 2.2.1.*

A neutral fermion initial, or final Fock state can be labeled by a strict partition

$$\langle\widehat{\mu}| = \alpha(-)^{r+|\widehat{\mu}|}\langle 0|\phi_{-m_{2r}} \ldots \phi_{-m_1} = \alpha(-)^{r+|\widehat{\mu}|}\langle 0|\overleftarrow{\prod_{j=1}^{2r}} \phi_{-m_j}$$

$$|\widehat{\mu}\rangle = \alpha(-)^{r}\phi_{m_1} \ldots \phi_{m_{2r}}|0\rangle = \alpha(-)^{r}\overrightarrow{\prod_{j=1}^{2r}} \phi_{m_j}|0\rangle \tag{2.2}$$

where $m_1 > \ldots > m_{2r} \geq 0$, $|\widehat{\mu}| = \sum_{j=1}^{2r} m_j$, $\alpha = 1$, for $m_{2r} \geq 1$, and $\alpha = \sqrt{2}$, for $m_{2r} = 0$. An arrow on a product indicates the direction in which the value of the index of that product increases.

2.3.1 Remark

In KP theory, positively and negatively charged fermion modes translate to distinct horizontal and distinct vertical parts. These combine, in a standard way, to form partitions that are not necessarily strict [3]. In BKP theory, there are only neutral modes, which translate to one set of distinct parts, which form strict partitions. This is why only strict partitions appear in this work.

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2.4 A Heisenberg sub-algebra

We refer the reader to [4] for complete definitions of the infinite dimensional Lie algebra B_∞, and its presentation in terms of bilinears in ϕ_m. Here, all we need is the Heisenberg sub-algebra generated by $\lambda_m \in B_\infty$, where

$$\lambda_m = \frac{1}{2}\sum_{j\in\mathbb{Z}}(-)^{j+1}\phi_j\phi_{-j-m}, \quad m \in \mathbb{Z}_{\text{odd}} \tag{2.3}$$

which satisfy the commutation relations

$$[\lambda_m, \lambda_n] = \frac{m}{2}\delta_{m+n,0}, \quad m, n \in \mathbb{Z}_{\text{odd}} \tag{2.4}$$

$$[\lambda_m, \phi_n] = \phi_{n-m}, \quad m \in \mathbb{Z}_{\text{odd}},\ n \in \mathbb{Z} \tag{2.5}$$

2.5 Evolution operators

Writing $\Lambda_\pm(\mathbf{x}_{\text{odd}}) = \sum_{m\in\pm\mathbb{N}_{\text{odd}}} x_m\lambda_m$, and $\zeta_\pm(\mathbf{x}_{\text{odd}}, k) = \sum_{m\in\pm\mathbb{N}_{\text{odd}}} x_m k^m$, a standard computation shows that

$$[\Lambda_\pm(\mathbf{x}_{\text{odd}}), \Phi(k)] = \zeta_\pm(\mathbf{x}_{\text{odd}}, k)\Phi(k) \tag{2.6}$$

which implies

$$e^{\Lambda_\pm(\mathbf{x}_{\text{odd}})}\Phi(k)e^{-\Lambda_\pm(\mathbf{x}_{\text{odd}})} = \Phi(k)e^{\zeta_\pm(\mathbf{x}_{\text{odd}},k)} \tag{2.7}$$

so that the operators $e^{\pm\Lambda_\pm(\mathbf{x}_{\text{odd}})}$ act as (forward and backward) evolution operators.

2.6 A choice of parameters

Setting $x_m = \frac{2}{m}z^{-m}$, $m \in \mathbb{Z}_{\text{odd}}$, and writing $\Lambda_\pm(\mathbf{x}_{\text{odd}}) = \Lambda_\pm(z)$, and $\zeta_\pm(\mathbf{x}_{\text{odd}}, k) = \zeta_\pm(z, k)$, we formally have

$$\zeta_+(z,k) = \sum_{m\in\mathbb{N}_{\text{odd}}}\frac{2}{m}\left(\frac{k}{z}\right)^m = \log\left(\frac{z+k}{z-k}\right) \tag{2.8}$$

$$\zeta_-(z,k) = -\sum_{m\in\mathbb{N}_{\text{odd}}}\frac{2}{m}\left(\frac{z}{k}\right)^m = \log\left(\frac{k-z}{k+z}\right)$$

2.7 Vertex operators

Consider the vertex operators

$$\Gamma_+^\phi(z) = e^{\Lambda_+(z)} = \exp\left(\sum_{m\in\mathbb{N}_{\text{odd}}}\frac{2}{m}z^{-m}\lambda_m\right) \tag{2.9}$$

$$\Gamma_-^\phi(z) = e^{-\Lambda_-(z)} = \exp\left(\sum_{m\in\mathbb{N}_{\text{odd}}}\frac{2}{m}z^{m}\lambda_{-m}\right) \tag{2.10}$$

Using equations (2.7) and (2.8), expanding and equating powers of k, we obtain

$$\Gamma_+^\phi(z)\phi_j\Gamma_+^\phi(-z) = \phi_j + 2\sum_{n=1}^{\infty}\frac{1}{z^n}\phi_{j-n} \tag{2.11}$$

$$\Gamma_-^\phi(-z)\phi_j\Gamma_-^\phi(z) = \phi_j + 2\sum_{n=1}^{\infty}(-z)^n\phi_{j+n} \tag{2.12}$$

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2.8 A commutation relation

Commuting two vertex operators

$$\begin{aligned}\Gamma^{\phi}_{+}(z)\Gamma^{\phi}_{-}(z') &= e^{\Lambda_+(z)}e^{-\Lambda_-(z')} = e^{[\Lambda_+(z),-\Lambda_-(z')]}\; e^{-\Lambda_-(z')}e^{\Lambda_+(z)} \\ &= e^{[\Lambda_+(z),-\Lambda_-(z')]}\; \Gamma^{\phi}_{-}(z')\Gamma^{\phi}_{+}(z)\end{aligned}$$

and using

$$\begin{aligned}\left[\Lambda_+(z), -\Lambda_-(z')\right] &= \sum_{m\in\mathbb{N}_{\text{odd}}}\sum_{n\in\mathbb{N}_{\text{odd}}} \frac{4}{mn} z^{-m}(z')^n[\lambda_m, \lambda_{-n}] \\ &= \sum_{m\in\mathbb{N}_{\text{odd}}}\sum_{n\in\mathbb{N}_{\text{odd}}} \frac{4}{mn} z^{-m}(z')^n \frac{m}{2}\delta_{m,n} \\ &= \sum_{m\in\mathbb{N}_{\text{odd}}} \frac{2}{m}\left(\frac{z'}{z}\right)^m = -\log\left(\frac{z-z'}{z+z'}\right)\end{aligned}$$

we obtain the basic commutation relation

$$\Gamma^{\phi}_{+}(z)\Gamma^{\phi}_{-}(z') = \left(\frac{z+z'}{z-z'}\right)\Gamma^{\phi}_{-}(z')\Gamma^{\phi}_{+}(z) \tag{2.13}$$

3. Interlacing strict partitions

In this section, we refer the reader to [1, 2] for the definition of interlacing partitions. We show how BKP vertex operators act on strict partitions to generate strict partitions that interlace with the initial ones. If $\widehat{\mu}$ and $\widehat{\nu}$ are interlacing, and $|\widehat{\mu}| \geq |\widehat{\nu}|$, where $|\widehat{\mu}|$ is the sum of the lengths of the parts of $\widehat{\mu}$, *etc*, we write $\widehat{\nu} \prec \widehat{\mu}$.

Lemma 1. *If $\widehat{\mu}$ and $\widehat{\nu}$ are strict partitions, as in Definition 3, then*

$$\langle\widehat{\nu}|\Gamma^{\phi}_{+}(z)|\widehat{\mu}\rangle = \begin{cases} 2^{n(\widehat{\nu}|\widehat{\mu})} z^{|\widehat{\nu}|-|\widehat{\mu}|}, & \widehat{\nu}\prec\widehat{\mu} \text{ and } n(\widehat{\nu}) = n(\widehat{\mu}) \\ (-)^{n(\widehat{\mu})}2^{n(\widehat{\nu}|\widehat{\mu})+\frac{1}{2}} z^{|\widehat{\nu}|-|\widehat{\mu}|}, & \widehat{\nu}\prec\widehat{\mu} \text{ and } n(\widehat{\nu}) = n(\widehat{\mu})-1 \\ 0, & \text{otherwise}\end{cases} \tag{3.1}$$

$$\langle\widehat{\mu}|\Gamma^{\phi}_{-}(z)|\widehat{\nu}\rangle = \begin{cases} 2^{n(\widehat{\nu}|\widehat{\mu})} z^{|\widehat{\mu}|-|\widehat{\nu}|}, & \widehat{\nu}\prec\widehat{\mu} \text{ and } n(\widehat{\nu}) = n(\widehat{\mu}) \\ (-)^{n(\widehat{\mu})}2^{n(\widehat{\nu}|\widehat{\mu})+\frac{1}{2}} z^{|\widehat{\mu}|-|\widehat{\nu}|}, & \widehat{\nu}\prec\widehat{\mu} \text{ and } n(\widehat{\nu}) = n(\widehat{\mu})-1 \\ 0, & \text{otherwise}\end{cases} \tag{3.2}$$

where $\{n(\widehat{\mu}), n(\widehat{\nu})\}$ is the number of non-zero parts in $\{\widehat{\mu}, \widehat{\nu}\}$, and $n(\widehat{\nu}|\widehat{\mu})$ is the number of non-zero parts in $\widehat{\nu}$ (the smaller partition), that are not in $\widehat{\mu}$ (the larger partition). It is important to notice that only $n(\widehat{\nu}|\widehat{\mu})$ appears in both of the above equations, and not $n(\widehat{\mu}|\widehat{\nu})$.

Proof. Setting $m_{2r+1} = -1$, we have

$$\begin{aligned}
\Gamma_+^{\phi}(z)|\widehat{\mu}\rangle &= \alpha(-)^r \Gamma_+^{\phi}(z) \overrightarrow{\prod_{j=1}^{2r}} \phi_{m_j}|0\rangle = \alpha(-)^r \overrightarrow{\prod_{j=1}^{2r}} \left(\Gamma_+^{\phi}(z)\phi_{m_j}\Gamma_+^{\phi}(-z)\right)|0\rangle \\
&= \alpha(-)^r \overrightarrow{\prod_{j=1}^{2r}} \left(\phi_{m_j} + 2\sum_{i=1}^{m_j - m_{j+1}-1} \frac{1}{z^i}\phi_{m_j - i} + \frac{1}{z^{m_j - m_{j+1}}}\phi_{m_{j+1}}\right)|0\rangle \\
&= \sum_{\substack{\widehat{\nu}\prec\widehat{\mu} \\ n(\widehat{\nu})=n(\widehat{\mu})}} 2^{n(\widehat{\nu}|\widehat{\mu})} z^{|\widehat{\nu}|-|\widehat{\mu}|}|\widehat{\nu}\rangle + (-)^{n(\widehat{\mu})}\sqrt{2} \sum_{\substack{\widehat{\nu}\prec\widehat{\mu} \\ n(\widehat{\nu})=n(\widehat{\mu})-1}} 2^{n(\widehat{\nu}|\widehat{\mu})} z^{|\widehat{\nu}|-|\widehat{\mu}|}|\widehat{\nu}\rangle
\end{aligned}$$

The proof of equation (3.2) goes along similar lines.

3.1 Condition 1 on BKP plane partitions

We can now see the origin of condition *1*, stated above. As the vertex operators act on a diagonal slice to form the subsequent diagonal slice, Lemma 1 says that every time a new path starts, we pick up a factor of 2. When a path ends, there is no such contribution. This follows from the fact that $n(\widehat{\nu}|\widehat{\mu})$ appears in *both* equations in Lemma 1, as mentioned above.

4. Diagonally strict plane partitions

In this section, we show how interlacing strict partitions, stacked vertically, form diagonally strict plane partitions with h-paths that are 2-coloured and 1-column wide.

Definition 4. *Assuming that the highest column of a plane partition is at the north west corner, as in the example in figure 1, a diagonally strict plane partition $\widehat{\pi}$ is a plane partition whose vertical slices, along all diagonals that run from north west to south east, are strict partitions.*

4.1 Condition 2 on BKP plane partitions

The diagonal strictness condition does not allow any 4 height-h, $h > 0$, columns to be in a 2×2 formation. Equivalently, every path is 1-column wide. Rather than give a formal, and definitely tedious proof of this simple observation, we encourage the reader to verify it by experimenting on a few simple examples.

4.2 Generating BKP plane partitions

Following the choice of parameters used in [6] and related papers, we consider the scalar product

$$\widehat{S}(q) = \langle 0| \overleftarrow{\prod_{j=1}^{\infty}} \Gamma_+^{\phi}\left(q^{\frac{-2j+1}{2}}\right) \overrightarrow{\prod_{k=1}^{\infty}} \Gamma_-^{\phi}\left(q^{\frac{2k-1}{2}}\right)|0\rangle \tag{4.1}$$

$$= \sum_{\widehat{\mu}} \langle 0| \overleftarrow{\prod_{j=1}^{\infty}} \Gamma_+^{\phi}\left(q^{\frac{-2j+1}{2}}\right)|\widehat{\mu}\rangle\langle\widehat{\mu}| \overrightarrow{\prod_{k=1}^{\infty}} \Gamma_-^{\phi}\left(q^{\frac{2k-1}{2}}\right)|0\rangle \tag{4.2}$$

where q is an indeterminate and $\sum_{\widehat{\mu}}$ indicates a sum over all strict partitions $\widehat{\mu}$. From Lemma 1, we know that $\Gamma_+^\phi(z)|\widehat{\mu}\rangle$ and $\langle\widehat{\mu}|\Gamma_-^\phi(z)$ generate all strict partitions $\widehat{\nu} \prec \widehat{\mu}$. As in [6], the arguments in the vertex operators are chosen such that the generated diagonally strict plane partitions are weighted by their volume. In addition, the vertex operators generate plane partitions with a multiplicity $2^{p(\widehat{\pi})}$, where $p(\widehat{\pi})$ is the total number of h-paths in $\widehat{\pi}$. It follows that equation (4.2) receives a contribution of

$$\prod_{j=1}^{M}\langle\widehat{\nu}_{-j}|\Gamma_+^\phi\left(q^{\frac{-2j+1}{2}}\right)|\widehat{\nu}_{-j+1}\rangle\prod_{k=1}^{N}\langle\widehat{\nu}_{k-1}|\Gamma_-^\phi\left(q^{\frac{2k-1}{2}}\right)|\widehat{\nu}_k\rangle = 2^{p(\widehat{\pi})}q^{|\widehat{\pi}|}$$

for each diagonally strict plane partition given by

$$\widehat{\pi} = \{\emptyset = \widehat{\nu}_{-M} \prec \ldots \prec \widehat{\nu}_{-1} \prec \widehat{\nu}_0 \succ \widehat{\nu}_1 \ldots \succ \widehat{\nu}_N = \emptyset\}$$

where $|\widehat{\pi}|$ is the volume of $\widehat{\pi}$. From that we conclude that

$$\widehat{S}(q) = \sum_{\widehat{\pi}} 2^{p(\widehat{\pi})}q^{|\widehat{\pi}|}$$

Applying the commutation relation equation (2.13) repeatedly to equation (4.1), one recovers a product form for the generating function $\widehat{S}(q)$ of plane partitions that satisfy the two conditions stated above:

$$\widehat{S}(q) = \prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^n \tag{4.3}$$

5. Conclusion and remarks

In hindsight, equation (4.3) is what we should have expected on the basis of the commutation relation of the vertex operators in equation (2.13). What is new is that equation (4.3) counts plane partitions that do *not* form a subset of the plane partitions counted by MacMahon's well-known result, re-derived in [1].

Clearly, it would be interesting to carry out a comprehensive study of plane partitions that are generated by various classes of free fermions. These would correspond to other integrable hierarchies, such as CKP, DKP, multicomponent hierarchies, such as n-KP, n-BKP, *etc*, and restricted versions of these, including KdV, *etc.* This is beyond the limited scope of this note, but we plan to report on them in further work.

An important question is whether the above result is relevant to topological string theory. Since the result of [2] relates KP theory, which is based on A_∞ to a topological string that is dual to U(N) Chern-Simons theory, in the limit $N \to \infty$, we naively expect that equation (4.3) is relevant to a topological string that is dual to $O(N)$ Chern-Simons theory, in the limit $N \to \infty$ [7].

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